

# Berry's Phase in the Presence of a Dissipative Medium

K.M. Fonseca Romero<sup>(1)</sup>, A.C. Aguiar Pinto<sup>(2)</sup>,  
and M. T. Thomaz<sup>(2)</sup>

<sup>(1)</sup> *Universidad Nacional, Facultad de Ciencias,  
Departamento de Física, Ciudad Universitaria,  
Bogotá, Colombia*

<sup>(2)</sup> *Instituto de Física - Univ. Federal Fluminense  
Av. Gal. Milton Tavares de Souza s/n.º,  
CEP: 24210-340, Niterói, R.J., Brazil*

## Abstract

We consider the spin 1/2 model coupled to a slowly varying magnetic field in the presence of a weak damping represented by a Lindblad-form operators. We show that Berry's geometrical phase remains unaltered by the two dissipation mechanism considered. Dissipation effects are twofold: a shrinking in the modulus of the Bloch's vector, which characterizes coherence loss and a time dependent (dissipation related) precession angle. We show that the line broadening of the Fourier transformation of the components of magnetization is only due to the presence of dissipation.

PACS numbers: 03.65.Bz, 03.65.-w, 05.30, 42.50.Lc

Keywords: Berry's Phase, Lindblad-form operator, Decoherence

# 1 Introduction

The existence of geometric phases in quantum systems, ever since its discovery by Berry [1], has attracted considerable interest both from the theoretical and experimental viewpoints. Several applications of this phenomenon in different areas of physics have also been studied[2, 3, 4]. Comparatively much less work has been devoted to the question of the dynamical evolution of such systems in the presence of a weakly dissipative medium. Basically the available results can be summarized as follows: nonhermitian operators lead to a modification of Berry's phase[5, 6, 7], stochastically evolving magnetic fields produce both energy shift and broadening[8], phenomenological weakly dissipative Liouvillians alter Berry's phase by introducing an imaginary correction[9] or causing damping and mixing of the density matrix elements[10]. Ellinas *et al.* [9] obtain their results studying the eigenmatrices of the complete Liouvillian superoperator in a time-independent basis, while Gamliel and Freed[10], find closed formal expressions for the density matrix representation in the instantaneous frame of the hamiltonian in the adiabatic limit, and in the weak dissipation approximation. However, even in this regime the results in the cases studied can only be extracted through approximations (in addition to the adiabatic approximation) or numerical computation.

In the present work, we consider the celebrated example of a spin  $1/2$  particle coupled to a slowly evolving magnetic field in the presence of a weakly dissipative medium as represented by a Lindblad form superoperator[11, 12, 13], incorporated in a convenient and physically motivated frame. In the absence of dissipation, a simple geometrical interpretation of the results emerges in terms of both the Bloch's vector[9] and the phase vector[14]. The geometrical phase appears as a "delay" or an "advance" in the precession period of the Bloch's vector with respect to the period dictated by the magnetic field's frequency. The precession of this vector occurs at a fixed angle with respect to a fixed axis about which the external magnetic field precesses, as usual. We will introduce the dissipation via a semi-group type dynamics, such that dissipation does not alter the precession frequency. However, the  $x$ ,  $y$  and  $z$  components of Bloch's vector are altered in different ways by nonunitary effects. In particular, the modulus of Bloch's vector will shrink. The relation between the reduction of Bloch's vector modulus and the loss of coherence has been explored by Stodolsky and collaborators[15, 16, 17]. In order to make it explicit, we introduce the linear entropy (or idempotency defect) as a measure of purity loss[18]

$$\delta(t) = 1 - \text{Tr } \rho^2(t) \quad (1)$$

and write

$$\rho(t) = \frac{1}{2} \begin{pmatrix} 1 + S_z(t) & S_x(t) - iS_y(t) \\ S_x(t) + iS_y(t) & 1 - S_z(t) \end{pmatrix} = \frac{1}{2}(\mathbb{1} + \vec{S}(t) \cdot \vec{\sigma}), \quad (2)$$

where  $\vec{S}(t)$  is the Bloch's vector. We get

$$\delta(t) = \frac{1 - |\vec{S}(t)|^2}{2}, \quad (3)$$

which confirms that the shrinking of the modulus of  $\vec{S}(t)$  is a measure of coherence loss.

In order to discuss dissipative effects on interference patterns due to Berry's geometrical phase the natural framework is to consider the evolution of the density matrix, which completely characterizes the interference effect due to the different geometric phases acquired by the eigenvectors of the hamiltonian of the system, even in the absence of dissipation[9]. Of course, this time we add a nonunitary Liouville operator contribution to the dynamics.

In section 2 we present the two level quantum system of a spin 1/2 in the presence of an external magnetic field precessing with constant angular velocity around a fixed axis. The master equation of the model is written in what we call the diagonal frame where Lindblad superoperators are introduced to describe the nonunitary part of the Liouville operator. In subsection 2.1 we consider the adiabatic limit of the quantum system in thermal equilibrium with a reservoir of electromagnetic fields and in subsection 2.2 we consider the case of a dephasing process. In both cases, we treat the coupling of the quantum system with its environment in the weak regime. In section 3 we show how the geometric phases and the dissipation effects acts on the  $z$  component of the magnetization of the spin 1/2 coupled to a reservoir of electromagnetic fields at thermal equilibrium. In section 4 we summarize our conclusions; finally, in appendix A, we present the non-unitary part of the liouvillians of the models under consideration in the instantaneous basis of the hamiltonian.

## 2 The adiabatic limit of the spin 1/2 model in the weak coupling regime

We consider a spin 1/2 variable (two level model) coupled to a time dependent magnetic field precessing around the z-axis. The unitary contribution for this evolution is given by the hamiltonian

$$\begin{aligned} H_s(t) &= \mu \vec{\sigma} \cdot \vec{B}(t) \\ &= \mu B \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\omega t} \\ \sin(\theta)e^{i\omega t} & -\cos(\theta) \end{pmatrix}, \end{aligned} \quad (4)$$

written in the basis of the eigenstates of the  $z$ -component of the spin, where  $B$  is the norm of the external magnetic field,  $\theta$  its azimuthal angle,  $\omega$  the precession frequency and the constant  $\mu = \frac{g\mu_B}{2}$ , being  $g$  Landé's factor and  $\mu_B$  the Bohr magneton. We are using natural units ( $c = \hbar = 1$ ). For the sake of later calculations, it is convenient to define two unitary transformations: the first one,  $R(\omega, t)$ , takes us to the rotating frame where the hamiltonian is no longer time dependent; the second one,  $D(B, \theta, \omega)$ , diagonalizes the effective hamiltonian (time independent) that drives the dynamics of the final matrix representation of the density operator. After the first transformation,  $R(\omega, t) = e^{-\frac{i\omega t}{2}\sigma_z}$ , the density matrix and the hamiltonian read

$$\rho_R(t) = e^{i\frac{\omega t}{2}\sigma_z} \rho(t) e^{-i\frac{\omega t}{2}\sigma_z}, \quad (5)$$

and

$$H_R = \mu B (\sin(\theta)\sigma_x + \cos(\theta)\sigma_z). \quad (6)$$

In analogous manner, after the second transformation we get, in the diagonal frame, the density matrix

$$\rho_D(t) = \mathbf{D}^T \rho_R(t) \mathbf{D}, \quad (7)$$

and the effective hamiltonian

$$\begin{aligned} H_D &= \mathbf{D}^T (H_R - \frac{\omega}{2}\sigma_z) \mathbf{D} \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}, \end{aligned} \quad (8)$$

where  $\lambda_1 = \sqrt{\mu^2 B^2 \sin^2(\theta) + (\mu B \cos(\theta) - \frac{\omega}{2})^2}$ . The rotation matrix  $\mathbf{D} = \mathbf{D}^T$  is equal to

$$\mathbf{D} = \sqrt{\frac{1}{2} - \frac{1}{2\lambda_1}(\mu B \cos(\theta) - \frac{\omega}{2})} \sigma_x + \sqrt{\frac{1}{2} + \frac{1}{2\lambda_1}(\mu B \cos(\theta) - \frac{\omega}{2})} \sigma_z. \quad (9)$$

One possible way to add dissipative contributions to the above dynamics is to include a Lindblad type superoperator in the evolution equation in the diagonal frame

$$\frac{d}{dt} \rho_D(t) = -i [\lambda_1 \sigma_z, \rho_D(t)] + k \mathcal{L}_D \rho_D(t), \quad (10)$$

where  $k$  is the dissipation constant. The weak coupling regime is characterized by the condition  $\frac{k}{\lambda_1} \ll 1$ .

Our aim is isolating, in the density matrix, the effects of dissipation on interference due to geometric phases. The representation of the density matrix in the diagonal frame is not very enlightening for this purpose. This is better realized in some basis of the instantaneous eigenvectors of hamiltonian (4), as was done by Gamliel and Freed [10]. We define  $\rho_I(t)$ , the matrix density in a basis of the instantaneous eigenvectors of hamiltonian (4)[19]. The relation between  $\rho_I(t)$  and  $\rho_D(t)$  is

$$\rho_I(t) = \mathbf{V}^\dagger(t) \mathbf{D} \rho_D(t) \mathbf{D} \mathbf{V}(t), \quad (11)$$

where the matrix  $\mathbf{V}(t)$  is equal to

$$\mathbf{V}(t) = \begin{pmatrix} \cos(\frac{\theta}{2})e^{-\frac{i\omega t}{2}} & -\sin(\frac{\theta}{2})e^{-\frac{i\omega t}{2}} \\ \sin(\frac{\theta}{2})e^{\frac{i\omega t}{2}} & \cos(\frac{\theta}{2})e^{\frac{i\omega t}{2}} \end{pmatrix}. \quad (12)$$

The time evolution of  $\rho_I(t)$  is given by

$$\frac{d}{dt} \rho_I(t) = -i \left[ \left( \mu B + \frac{\omega}{2} \right) \sigma_z - \frac{\omega}{2} \sigma_n(t), \rho_I(t) \right] + k \mathcal{L}_I \rho_I(t) \quad (13a)$$

where

$$\sigma_n(t) = \begin{pmatrix} \cos(\theta) & -\sin(\theta)e^{-i\omega t} \\ -\sin(\theta)e^{i\omega t} & -\cos(\theta) \end{pmatrix} \quad (13b)$$

and  $\mathcal{L}_I \rho_I(t)$  is obtained from  $\mathcal{L}_D \rho_D(t)$  through a similarity transformation equivalent to (11).

Before we specialize our discussion to any particular liouvillian, we study the adiabatic limit of eqs.(13) for the coupling constant  $k$  in the weak regime. In this regime, the matrix  $\mathcal{L}_I \rho_I(t)$  is written as a linear superposition of the elements  $\rho_{ij}^I(t)$ . We remind that the density matrix of a two level model must satisfy two conditions: *i*)  $Tr(\rho_I(t)) = 1$  and *ii*)  $\rho_{21}^I(t) = (\rho_{12}^I(t))^*$ . As a consequence of those conditions, the density matrix has only two independent elements. We take the elements  $\rho_{11}^I(t)$  and  $\rho_{12}^I(t)$  as our two independent entries. The general form for the time equations of those two elements in the weak coupling regime is

$$\frac{d}{dt} \rho_{11}^I(t) = a_{11}(\omega, k; t) \rho_{11}^I(t) + a_{12}(\omega, k; t) \rho_{12}^I(t) + a_{13}(\omega, k; t) \rho_{21}^I(t) + b_1(\omega, k; t) \quad (14a)$$

and

$$\begin{aligned} \frac{d}{dt} \rho_{12}^I(t) = & a_{21}(\omega, k; t) \rho_{11}^I(t) + (-2i\mu B + a_{22}(\omega, k; t)) \rho_{12}^I(t) + a_{23}(\omega, k; t) \rho_{21}^I(t) + \\ & + b_2(\omega, k; t). \end{aligned} \quad (14b)$$

We make the change of variables:

$$\rho_{ij}^I(t) \equiv e^{-i(E_i - E_j)t} \tilde{\rho}_{ij}(t) \quad (15a)$$

with  $E_1 = \mu B$  and  $E_2 = -\mu B$ . We may disclose the time scale  $T$  in the differential equations by introducing the following transformation upon the time parameter[20, 21]:

$$s \equiv \frac{t}{T}, \quad (15b)$$

where  $T = \frac{2\pi}{\omega}$ .

With the new variables, eqs.(14) become

$$\begin{aligned} \frac{d}{ds} \tilde{\rho}_{11}(s) = & T a_{11}(\omega, k; s) \tilde{\rho}_{11}(s) + T a_{12}(\omega, k; s) e^{-2i\mu B T s} \tilde{\rho}_{12}(s) + \\ & + T a_{13}(\omega, k; s) e^{2i\mu B T s} \tilde{\rho}_{21}(s) + T b_1(\omega, k; s), \end{aligned} \quad (16a)$$

$$\begin{aligned} \frac{d}{ds} \tilde{\rho}_{12}(s) = & T a_{21}(\omega, k; s) e^{2i\mu B T s} \tilde{\rho}_{11}(s) + T a_{22}(\omega, k; s) \tilde{\rho}_{12}(s) + \\ & + T a_{23}(\omega, k; s) e^{4i\mu B T s} \tilde{\rho}_{21}(s) + T b_2(\omega, k; s) e^{2i\mu B T s}. \end{aligned} \quad (16b)$$

We take the differential equation for  $\tilde{\rho}_{12}(t)$  to exemplify the discussion of the adiabatic limit of eqs.(16). At this point we will follow closely the references [20, 21]. We point out that the adiabatic approximation is not recovered by an  $\omega$  expansion of the terms on the r.h.s. of eqs.(16).

The Volterra equation[22] obtained from eq.(16b) is

$$\begin{aligned}\tilde{\rho}_{12}(s) &= \tilde{\rho}_{12}(0) + T \int_0^s a_{22}(\omega, k; s') \tilde{\rho}_{12}(s') ds' + T \int_0^s a_{21}(\omega, k; s') e^{2i\mu B T s'} \tilde{\rho}_{11}(s') ds' + \\ &+ T \int_0^s a_{23}(\omega, k; s') e^{4i\mu B T s'} \tilde{\rho}_{21}(s') ds' + T \int_0^s b_2(\omega, k; s') e^{2i\mu B T s'} ds'.\end{aligned}\quad (17)$$

In the limit  $T \rightarrow \infty$ , the Riemann-Lebesgue Theorem [23] gives that

$$\lim_{T \rightarrow \infty} \int_0^s F(s') e^{i\alpha\mu T B s'} ds' = 0, \quad (18)$$

if  $F(s')$  is a piece-wise continuous function in the interval  $[0, s]$  and  $\alpha \in \mathbb{R}$ . As a consequence of this theorem the last three terms on the r.h.s. of eq.(17) vanish in the adiabatic limit ( $\frac{\omega}{\mu B} \rightarrow 0 \Rightarrow T \rightarrow \infty$ ).

Integrating by parts eq.(18) we get

$$\int_0^s F(s') e^{i\alpha\mu T B s'} ds' = \frac{1}{iT} \frac{1}{\alpha\mu B} [F(s) e^{i\alpha\mu T B s} - F(0)] - \frac{1}{iT} \frac{1}{\alpha\mu B} \int_0^s \frac{d}{ds'} (F(s')) e^{i\alpha\mu T B s'} ds'. \quad (19)$$

From eqs.(13) and (14b), the coefficients  $a_{2j}$  and  $b_2$  have the general dependence on  $\omega$  and  $k$ :

$$a_{21}(\omega, k; s) = \omega a_{21}^{(0)}(s) + k \left( \tilde{a}_{21}^{(0)}(s) + \frac{\omega}{\mu B} A_{21}(s) + \mathcal{O} \left( \left( \frac{\omega}{\mu B} \right)^2 \right) \right), \quad (20a)$$

$$a_{22}(\omega, k; s) = -i\omega(1 - \cos(\theta)) + k \left( \tilde{a}_{22}^{(0)}(s) + \frac{\omega}{\mu B} A_{22}(s) + \mathcal{O} \left( \left( \frac{\omega}{\mu B} \right)^2 \right) \right), \quad (20b)$$

$$a_{23}(\omega, k; s) = k \left( \tilde{a}_{23}^{(0)}(s) + \frac{\omega}{\mu B} A_{23}(s) + \mathcal{O} \left( \left( \frac{\omega}{\mu B} \right)^2 \right) \right) \quad (20c)$$

and

$$b_2(\omega, k; s) = \omega b_2^{(0)}(s) + k \left( \tilde{b}_2^{(0)}(s) + \frac{\omega}{\mu B} B_2(s) + \mathcal{O} \left( \left( \frac{\omega}{\mu B} \right)^2 \right) \right). \quad (20d)$$

The integrands of all integrals on the r.h.s. of eq.(17) which contain the oscillatory function  $e^{i\alpha\mu B T s'}$ , with  $\alpha = 2$  or  $4$ , after integration by parts, each of those integrals has its order in  $T$  decreased by one unit, and acquires a multiplying constant of value  $\frac{k}{\mu B}$  or  $\frac{\omega}{\mu B}$ . From the conditions satisfied by the two simultaneous regimes: *i*) adiabatic limit ( $\frac{\omega}{\mu B} \ll 1$ ) and *ii*) weak coupling limit ( $\frac{k}{\mu B} \ll 1$ ), we can neglect those terms in comparison to the first two terms on the r.h.s. of eq.(17). The two previous inequalities do not impose any constraint to the ratio  $\frac{k}{\omega}$ , though. The differential equation satisfied by  $\tilde{\rho}_{12}(s)$  in the adiabatic limit and weak coupling regime is

$$\frac{d}{ds}\tilde{\rho}_{12}(s) = T \left[ -i\omega(1 - \cos(\theta)) + k\tilde{a}_{22}^{(0)}(s) \right] \tilde{\rho}_{12}(s). \quad (21)$$

The term proportional to  $A_{22}(s)$  was dropped, since it is of higher order in  $\left(\frac{\omega}{\mu B}\right)$ . For  $k \gg \omega$ , the off-diagonal elements of the density matrix vanish before the external magnetic field  $\vec{B}(t)$  returns to its configuration at  $t = 0$ . The other uninteresting situation from the point of view of the appearance of an imaginary correction to the geometric phase is the condition  $k \ll \omega$  when the dissipation effects can still be neglected after one period. In this work we discuss the case  $k \sim \omega$  when both terms on the r.h.s. of eq.(21) contribute to the time evolution of  $\tilde{\rho}_{12}(s)$ .

By a similar discussion we obtain the time equation of  $\tilde{\rho}_{11}(s)$  in the adiabatic and weak coupling regimes,

$$\frac{d}{ds}\tilde{\rho}_{11}(s) = Tk\tilde{a}_{11}^{(0)}(s)\tilde{\rho}_{11}(s) + T \left[ \omega b_1^{(0)}(s) + k\tilde{b}_1^{(0)}(s) \right]. \quad (22a)$$

From eqs.(13) we can affirm that

$$a_{11}(\omega, k; s) = k \left( \tilde{a}_{11}^{(0)}(s) + \frac{\omega}{\mu B} A_{11}(s) + \mathcal{O} \left( \left( \frac{\omega}{\mu B} \right)^2 \right) \right), \quad (22b)$$

$$b_1(\omega, k; s) = k \left( \tilde{b}_1^{(0)}(s) + \frac{\omega}{\mu B} B_1(s) + \mathcal{O} \left( \left( \frac{\omega}{\mu B} \right)^2 \right) \right). \quad (22c)$$

In order to verify if we can get imaginary phases from eqs. (21) and (22a) due to the coupling of the quantum system to a dissipative medium, we write the density operator at  $t = 0$  in the instantaneous basis of hamiltonian (4)



$$\rho(0) = \sum_{j,l=1}^2 \alpha_{jl}(0) |\phi_j^0\rangle \langle \phi_l^0|. \quad (23)$$

We have  $\mathbf{H}(0)|\phi_j^0\rangle = E_j(0)|\phi_j^0\rangle$ ,  $j = 1$  and  $2$ . Being  $|\phi_j^0(t)\rangle$  the time evolution of the eigenvector  $|\phi_j^0\rangle$ , we have

$$\rho(t) = \sum_{j,l=1}^2 \alpha_{jl}(t) |\phi_j^0(t)\rangle \langle \phi_l^0(t)|, \quad (24)$$

where the time evolution of  $|\phi_j^0(t)\rangle$  is driven by  $\mathbf{H}(t)$ . Differently from Gamliel and Freed, we include the phases coming from the unitary evolution and the geometric phase in the dyadic product  $|\phi_j^0(t)\rangle \langle \phi_l^0(t)|$ . In the adiabatic approximation, we get

$$|\phi_j^0(t)\rangle = e^{i\gamma_j(t)} e^{-i\langle E_j(t) \rangle t} |\phi_j; t\rangle, \quad (25)$$

where  $\gamma_j(t)$  is the geometric phase acquired by the eigenvector  $|\phi_j^0\rangle$ ,  $|\phi_j; t\rangle$  is the instantaneous eigenvector of  $\mathbf{H}(t)$  ( $\mathbf{H}(t)|\phi_j; t\rangle = E_j(t)|\phi_j; t\rangle$ ) and  $\langle E_j(t) \rangle \equiv \frac{1}{t} \int_0^t dt' E_j(t')$ .

The density operator at any time, in the adiabatic limit is

$$\rho(t) = \sum_{j,l=1}^2 \alpha_{jl}(t) e^{i(\gamma_j(t) - \gamma_l(t))} e^{-i(\langle E_j(t) \rangle - \langle E_l(t) \rangle)t} |\phi_j; t\rangle \langle \phi_l; t|. \quad (26)$$

From eq.(26) we recognize that the phase  $e^{i(\gamma_j(t) - \gamma_l(t))}$  in the element  $\rho_{jl}^I(t)$  is just the difference of the geometric phases of the instantaneous eigenstates  $|\phi_j; t\rangle$  and  $|\phi_l; t\rangle$  in the absence of dissipation.

The dynamics of the coefficients  $\alpha_{jl}(t)$  is ruled by the nonunitary evolution of the quantum system and it is independent of the particular choice for the instantaneous eigenstates of the hamiltonian, up to a multiplicative constant[24]. In the model under consideration (see hamiltonian (4)), the eigenvalues  $E_j(t)$ ,  $j = 1$  and  $2$ , are time-independent. It is simple to get the time equations of  $\alpha_{11}(t)$  and  $\alpha_{12}(t)$  from eqs.(22a) and (21), respectively

$$\frac{d}{dt} \alpha_{11}(t) = k \left( \tilde{a}_{11}^{(0)}(t) \alpha_{11}(t) + \tilde{b}_1^{(0)}(t) \right), \quad (27a)$$

$$\frac{d}{dt} \alpha_{12}(t) = k \tilde{a}_{22}^{(0)}(t) \alpha_{12}(t). \quad (27b)$$

The constants  $\tilde{a}_{11}^{(0)}(t)$ ,  $\tilde{a}_{22}^{(0)}(t)$  and  $\tilde{b}_1^{(0)}(t)$  depend on the particular master equation that describes the behaviour of the quantum system interacting with the dissipative medium. In the next sub-sections we consider two particular interactions of the two level model with a reservoir: *i*) two level model in thermal equilibrium with a reservoir of electromagnetic fields; *ii*) dephasing process in a two level model.

## 2.1 Adiabatic limit of a two level model in thermal equilibrium

As discussed before, to incorporate the dissipative effects in the two level model we introduce the Lindblad superoperator in the diagonal frame. The master equation of the spin 1/2 model coupled to a reservoir of electromagnetic fields in thermal equilibrium in the diagonal frame is [11, 25]

$$\begin{aligned} \frac{d}{dt}\rho_D(t) = & -i[\lambda_1\sigma_z, \rho_D(t)] + k(\bar{n}+1)(2\sigma_-\rho_D(t)\sigma_+ - \rho_D(t)\sigma_+\sigma_- - \sigma_+\sigma_-\rho_D(t)) + \\ & + k\bar{n}(2\sigma_+\rho_D(t)\sigma_- - \rho_D(t)\sigma_-\sigma_+ - \sigma_-\sigma_+\rho_D(t)), \end{aligned} \quad (28)$$

where  $k$  is the dissipation constant at zero temperature and  $\bar{n}$  is the average number of excitations of the weakly coupled thermal oscillators at inverse temperature  $\beta$ . An important requirement for the introduction of this Lindblad type superoperator is that it leads asymptotically to a thermal equilibrium. In appendix A we give the master equation of this physical process in the instantaneous basis of hamiltonian (4) for arbitrary value of  $\omega$ .

From the master equation in the instantaneous basis of the hamiltonian we obtain the equations for  $\alpha_{11}(t)$  and  $\alpha_{12}(t)$ . These equations in the adiabatic and in the weak coupling limits become

$$\frac{d}{dt}\alpha_{11}(t) = -2k(1+2\bar{n})\alpha_{11}(t) + 2k\bar{n}, \quad (29a)$$

$$\frac{d}{dt}\alpha_{12}(t) = -k(1+2\bar{n})\alpha_{12}(t). \quad (29b)$$

The constant  $k$  does not come up on the r.h.s. of eqs.(29) due to the time variation of any classical parameter that characterizes the reservoir. From its explicit definition [11, 25] it can not be written as:  $f(t)\dot{g}(t)$ , where  $f(t)$  and  $g(t)$  are two regular time dependent functions. The same is true for  $\bar{n}$ . Therefore the imaginary phase  $\chi(t)$  defined as:  $\alpha_{12}(t) \equiv \alpha_{12}(0)e^{i\chi(t)}$ , with

$$\chi(t) = i \int_0^t k(1 + 2\bar{n}) dt' \quad (30)$$

is not geometric.

The solution of eq.(29a) is

$$\alpha_{11}(t) = \frac{\bar{n}}{1 + 2\bar{n}} + \left[ \alpha_{11}(0) - \frac{\bar{n}}{1 + 2\bar{n}} \right] e^{-2k \int_0^t (1 + 2\bar{n}) dt'}. \quad (31)$$

The exponential decay on the r.h.s. of eq.(31) means that the population of the instantaneous eigenstates of hamiltonian (4) varies in time and consequently the Adiabatic Theorem is not valid for this dissipation mechanism.

Exactly soluble models are always important checks to approximation schemes. Eq.(28) is exactly solved and the solutions are

$$\rho_{11}^D(t) = \frac{\bar{n}}{2\bar{n} + 1} \left[ 1 - e^{-2k(2\bar{n}+1)t} \right] + \rho_{11}^D(0) e^{-2k(2\bar{n}+1)t}, \quad (32a)$$

$$\rho_{12}^D(t) = \rho_{12}^D(0) e^{-(2i\lambda_1 + k(2\bar{n}+1))t}. \quad (32b)$$

It is straightforward to obtain the adiabatic and weak coupling limit eqs.(29) from eqs.(32).

## 2.2 Dephasing process in two level system

Another interesting process well studied in the standard textbooks[11, 25] is the phase destroying process which might appear due to elastic collisions. In general, those effects are incorporated in the master equation of the two level model, besides the energy dissipation process studied in subsection 2.1. Since we are studying the coupling of the spin 1/2 to a dissipative medium in the weak coupling limit, the inclusion of the phase destroying process in eq.(28) gives corrections to the coefficients  $a_{ij}(\omega, k; t)$  in eqs.(29). Due to the linearity of the equations, the new imaginary phases coming from the dephasing process are added to the ones obtained previously. For the sake of simplicity we study the imaginary phases acquired by the variables  $\alpha_{11}(t)$  and  $\alpha_{12}(t)$  only due to the dephasing process. The master equation written in the diagonal frame is

$$\frac{d}{dt}\rho_D(t) = -i[\lambda_1\sigma_z, \rho_D(t)] + \frac{k}{2}(\sigma_z\rho_D(t)\sigma_z - \rho_D(t)). \quad (33)$$

In appendix A we present the master equation of the spin 1/2 with the dephasing effect included for arbitrary value of the angular velocity  $\omega$  of the external magnetic field. In this subsection we study the time equations of coefficients  $\alpha_{11}(t)$  and  $\alpha_{12}(t)$  in the adiabatic and weak coupling regimes. Taking into account our discussion in section 2, eqs.(A.2) and (27) we obtain

$$\frac{d}{dt}\alpha_{11}(t) = 0, \quad (34a)$$

$$\frac{d}{dt}\alpha_{12}(t) = -k\alpha_{12}(t). \quad (34b)$$

that have the solutions:

$$\alpha_{11}(t) = \alpha_{11}(0), \quad (35a)$$

$$\alpha_{12}(t) = \alpha_{12}(0) e^{-\int_0^t k dt'}. \quad (35b)$$

From eq.(35a) we conclude that the Adiabatic Theorem is valid in this process, since the population at each quantum state does not vary along the adiabatic process. By analogous reasons to the ones discussed in subsection 2.1, the phase in eq.(35b) is not geometric but a time dependent imaginary phase that destroys the off-diagonal elements of the density matrix. Eq.(33) is exactly solved. The solutions (35) are easily recovered from the exact solutions when we calculate them in the adiabatic and weak coupling limits.

In order to understand why references [5]-[7] give an imaginary correction to the geometric phase and the models studied here do not, we compare eq.(2.3) of reference [5] with our eqs.(30) and (35b). The coefficient that multiplies of variable  $C_\lambda(t)$  in eq.(2.3) of reference [5] has the form  $\langle\theta(t)|\frac{d}{dt}|\psi(t)\rangle$ . Since the vector states depend on a periodic external parameter  $R(t)$ , this coefficient, in the adiabatic approximation, corresponds to a correction to Berry's phase written as a closed curve in parameter space. The coefficient  $k(1 + 2\bar{n})$  that multiplies  $\alpha_{12}(t)$  in eq.(29b) does not arise from any variation of an external parameter. The same is true for the coefficient  $k$  in eq.(34b). That is the reason that allows us to claim that the exponentials acquired from eqs.(30) and (35b) have no geometric origin, which means that the suppression terms are a function of time instead of some path parameter.

### 3 Contribution of the geometric phase to the magnetization

In order to verify how the geometric phases and the dissipation affect the physical quantities, we return to the spin 1/2 model in the presence of a reservoir of electromagnetic fields in thermal equilibrium. Under the initial condition  $\rho(0) = |\psi(0)\rangle\langle\psi(0)|$ , with  $|\psi(0)\rangle = \cos(\alpha)|+\rangle + \sin(\alpha)|-\rangle$ , in the adiabatic limit and weak coupling limit we end up with

$$\rho_I^{11}(t) = \frac{\bar{n}}{2\bar{n}+1} + \frac{1}{2} \left( \frac{1}{2\bar{n}+1} + \cos(\theta - 2\alpha) \right) e^{-2k(2\bar{n}+1)t}, \quad (36a)$$

$$\rho_I^{12}(t) = \frac{1}{2} \sin(2\alpha - \theta) e^{-2i\mu B t} e^{-k(2\bar{n}+1)t} e^{-ia\omega(1-\cos(\theta))t}. \quad (36b)$$

We have introduced the Berry's phase tracer  $a$  that helps us identify the contribution of Berry's phase to physical quantities. At the end, we take the tracer equal to one.

Thus we see that, in this model, dissipation does not affect Berry's geometrical phase, but only makes it harder to observe their interference effect: such information is contained in the off-diagonal terms of the density matrix in the instantaneous frame, which vanish. By the way, we can observe that the system considered in this contribution is analogous to the case of the classical Foucault pendulum where dissipation diminishes the amplitude, but does not affect the rotation of the oscillation plane.

The definition of the geometrical phase can also be given in terms of a phase vector as in reference[14], where the discussion is confined to pure states. It is, however, a relatively simple matter to extend the definition of the phase vector for mixed states. In this case we find that the loss of coherence will shorten the phase vector in a manner which is completely analogous to what happens to Bloch's vector. This norm reduction, as in the present case, reflects the asymptotic vanishing of off-diagonal density matrix elements[26].

Since we are interested in interference effects due to geometric phases, Bloch's vector, defined in eq.(2), is suitable to provide a graphic visualization of the density matrix [9]. From eq.(3), we have that these effects are clearly described by Bloch's vector in the instantaneous frame. In this frame the projection of Bloch's vector sweeps the  $xy$  plane and makes an angle smaller than that of the magnetic field by an amount proportional to the solid angle  $\Omega(\theta)$ , while its length decreases exponentially with a time rate of  $k(2\bar{n}+1)$ . We illustrate in figure 1 the shrinking of the projection of Bloch's vector in the  $xy$  plane, due to the presence of dissipation. Even though we have discussed the case  $k \sim \omega$ ,

we choose  $k/\omega = 10$  in order to show more clearly the decreasing of this projection in an interval  $4\pi/\mu B$ . Shortening in  $z$  is faster than in the  $xy$  plane, causing a time dependent azimuthal angle. Figure 2 shows the plot of the time evolution of Bloch's vector for  $k/\omega = 0.2$ . For  $t \rightarrow \infty$  Bloch's vector has only non-zero  $z$ -component and  $S_z(t \rightarrow \infty) = -\frac{1}{2\bar{n}+1}$ .

For time  $t$ , such that  $t \neq nT$  ( $T = 2\pi/\omega$  and  $n$  is an integer) the off-diagonal elements of the density matrix  $\rho_I(t)$  depends on the chosen condition satisfied by  $\langle \phi_i; t | \frac{d}{dt} | \phi_i; t \rangle$  [19]. For the sake of comparison with experiments it is necessary to calculate dissipation and geometrical phases effects on measurable quantities. In this model the natural candidates are the components of the magnetization vector  $\langle \vec{m} \rangle(t)$ . Let us consider the  $z$ -component of magnetization whose average value has the expression  $\text{Tr}(\rho(t)\mathbf{m}_z)$ . Since the trace is independent of the particular basis applied to calculate it the result is independent of our particular choice of  $\langle \phi_i; t | \frac{d}{dt} | \phi_i; t \rangle$ . In the adiabatic approximation and the weak coupling limit, using eqs.(36), we get the Fourier transform of  $\langle m_z \rangle(t)$ ,

$$\begin{aligned} \langle \tilde{m}_z \rangle(\omega') &= \frac{\mu}{\sqrt{2\pi}} \left[ \frac{\cos(\theta)}{2\bar{n}+1} \pi \delta(\omega') - i \cos(\theta) \left( \frac{1}{2\bar{n}+1} + \cos(\theta - 2\alpha) \right) \frac{1}{\omega' + 2ki(2\bar{n}+1)} + \right. \\ &\quad \left. -i \frac{1}{2} \sin(\theta) \sin(\theta - 2\alpha) \left( \frac{1}{\omega' + \Gamma + ki(2\bar{n}+1)} + \frac{1}{\omega' - \Gamma + ki(2\bar{n}+1)} \right) \right], \end{aligned} \quad (37a)$$

where  $\alpha$  is given by the initial condition, and the resonant frequency  $\Gamma$  is equal to

$$\Gamma \equiv 2\mu B - a\omega \cos(\theta). \quad (37b)$$

The first term of eq.(37a) corresponds to the constant component of the magnetic field, and the second one shows the dissipation effects on this field component. The last term on the r.h.s. of the above equation displays a real frequency shift which contains the contribution of the geometrical phase, as can be seen from eq.(37b), due the presence of the tracer  $a$ , and a line broadening caused only by the dissipative evolution. These effects on the magnetization agree with the ones derived in reference [8] where the path integral formalism was applied.

The expressions for the other components of magnetization are analogous, but somewhat lengthy.

## 4 Conclusions

In summary, we have presented an analytical solution of the adiabatic limit of a spin 1/2 in a precessing magnetic field embedded in a weakly dissipative medium, introduced phenomenologically. We consider two distinct nonunitary contributions that were accounted for by a Lindblad type superoperator in the diagonal frame. We are able to derive analytical expressions for the geometric and imaginary phases in both cases in the presence of a weak dissipation in the adiabatic limit without further approximations.

From eqs.(30) and (35b) we get that the nature (path dependent or time dependent) of the imaginary phase acquired by  $\alpha_{12}(t)$  depends on the mechanism that introduces the dissipation in the quantum system. In both cases that we have studied, the dissipation is present due to the two level system being in contact with a reservoir. The constant  $k$  on the r.h.s. of eq.(30) is associated to the time rate of population and not due to the variation of any external parameter. Consequently the quantum geometric phase for  $k = 0$  is not modified by a complex value. An analogous argument is valid to explain why the imaginary phase acquired by the entry  $\rho_{12}^I(t)$  in the dephasing process is not geometric, either. In this last model the Adiabatic Theorem continues to be true while it is not true anymore for the spin 1/2 coupled to the electromagnetic fields at thermal equilibrium. Differently from Ellinas *et al.* in reference[9] we do not call this complex phase as Berry's phase. We reserve the name of "Berry's phase" only to phases (real or imaginary) that are path dependent.

Decoherence effects are present and their manifestation is the shortening of the three components of the Bloch's vector. The fact that the dissipation effect causes the suppression of interference patterns due to the geometric phase is not a particular result for the chosen liouvillian. It is rather general, stemming from the fact that the dissipation mechanism is not related to the variation of any set of external periodic parameters.

## Acknowledgements

The authors are in debt with M.C. Nemes and J.G. Peixoto de Faria for useful discussions. The authors thank the referee for bringing to their attention the interesting model discussed in section 2.2. A.C.A.P. and M.T.T. thank E.V. Corrêa Silva for the careful reading of the manuscript. A.C. Aguiar Pinto thanks CNPq for financial support. M.T. Thomaz thanks CNPq for partial financial support.

## Appendix A: Master equations in the instantaneous basis of the hamiltonian

The master equation (28) in the instantaneous basis of the hamiltonian for arbitrary value of  $\omega$  is

$$\frac{d}{dt}\rho_I(t) = -i \left[ \left( \mu B + \frac{\omega}{2} \right) \sigma_z - \frac{\omega}{2} \sigma_n(t), \rho_I(t) \right] + k \mathcal{L}_I \rho_I(t) \quad (\text{A.1a})$$

where  $\sigma_n(t)$  is given by eq.(13b) and

$$\begin{aligned} \mathcal{L}_I \rho_I(t) = & \frac{2\bar{n}+1}{2} \left\{ -2\rho_I(t) + (1-\Lambda^2) \left[ \sigma_n(t)\rho_I(t)\sigma_n(t) - e^{-2i\omega t} \sigma_+(t)\rho_I(t)\sigma_+(t) - \right. \right. \\ & - e^{2i\omega t} \sigma_-(t)\rho_I(t)\sigma_-(t) \left. \right] + (1+\Lambda^2) \left[ \sigma_+(t)\rho_I(t)\sigma_-(t) + \sigma_-(t)\rho_I(t)\sigma_+(t) \right] - \\ & - \Lambda\sqrt{1-\Lambda^2} \left[ e^{i\omega t} \left( \sigma_n(t)\rho_I(t)\sigma_-(t) + \sigma_-(t)\rho_I(t)\sigma_n(t) \right) + \right. \\ & + e^{-i\omega t} \left( \sigma_+(t)\rho_I(t)\sigma_n(t) + \sigma_n(t)\rho_I(t)\sigma_+(t) \right) \left. \right] \left. \right\} - \\ & - \frac{1}{2} \left\{ \left\{ \rho_I(t), \Lambda\sigma_n(t) + \sqrt{1-\Lambda^2}(e^{-i\omega t}\sigma_+(t) + e^{i\omega t}\sigma_-(t)) \right\} + 2\Lambda \left[ \sigma_+(t)\rho_I(t)\sigma_-(t) - \right. \right. \\ & - \sigma_-(t)\rho_I(t)\sigma_+(t) \left. \right] + \sqrt{1-\Lambda^2} \left[ e^{-i\omega t} \left( \sigma_n(t)\rho_I(t)\sigma_+(t) - \sigma_+(t)\rho_I(t)\sigma_n(t) \right) + \right. \\ & + e^{i\omega t} \left( \sigma_-(t)\rho_I(t)\sigma_n(t) - \sigma_n(t)\rho_I(t)\sigma_-(t) \right) \left. \right] \left. \right\}. \end{aligned} \quad (\text{A.1b})$$

In eq.(A.1b), we define:  $\Lambda \equiv \frac{1}{\lambda_1} \left( \mu B \cos(\theta) - \frac{\omega}{2} \right)$  and

$$\sigma_+(t) = e^{i\omega t} \left[ \frac{\sin(\theta)}{2} \sigma_z + \left( \cos\left(\frac{\theta}{2}\right) \right)^2 e^{-i\omega t} \sigma_+ - \left( \sin\left(\frac{\theta}{2}\right) \right)^2 e^{i\omega t} \sigma_- \right] \quad (\text{A.1c})$$

and  $\sigma_-(t) \equiv \left( \sigma_+(t) \right)^\dagger$ .

The master equation (33) in the instantaneous basis of the hamiltonian for arbitrary value of  $\omega$  is



$$\frac{d}{dt}\rho_I(t) = -i \left[ \left( \mu B + \frac{\omega}{2} \right) \sigma_z - \frac{\omega}{2} \sigma_n(t), \rho_I(t) \right] + \frac{k}{2} \mathcal{L}_I \rho_I(t) \quad (\text{A.2a})$$

where

$$\begin{aligned} \mathcal{L}_I \rho_I(t) = & \Lambda^2 \sigma_n(t) \rho_I(t) \sigma_n(t) + \Lambda \sqrt{1 - \Lambda^2} \left[ e^{-i\omega t} \left( \sigma_n(t) \rho_I(t) \sigma_+(t) + \sigma_+(t) \rho_I(t) \sigma_n(t) \right) + \right. \\ & + e^{i\omega t} \left( \sigma_n(t) \rho_I(t) \sigma_-(t) + \sigma_-(t) \rho_I(t) \sigma_n(t) \right) \left. \right] + (1 - \Lambda^2) \left[ e^{-2i\omega t} \sigma_+(t) \rho_I(t) \sigma_+(t) + \right. \\ & + e^{2i\omega t} \sigma_-(t) \rho_I(t) \sigma_-(t) + \sigma_+(t) \rho_I(t) \sigma_-(t) + \sigma_-(t) \rho_I(t) \sigma_+(t) \left. \right] - \rho_I(t). \quad (\text{A.2b}) \end{aligned}$$

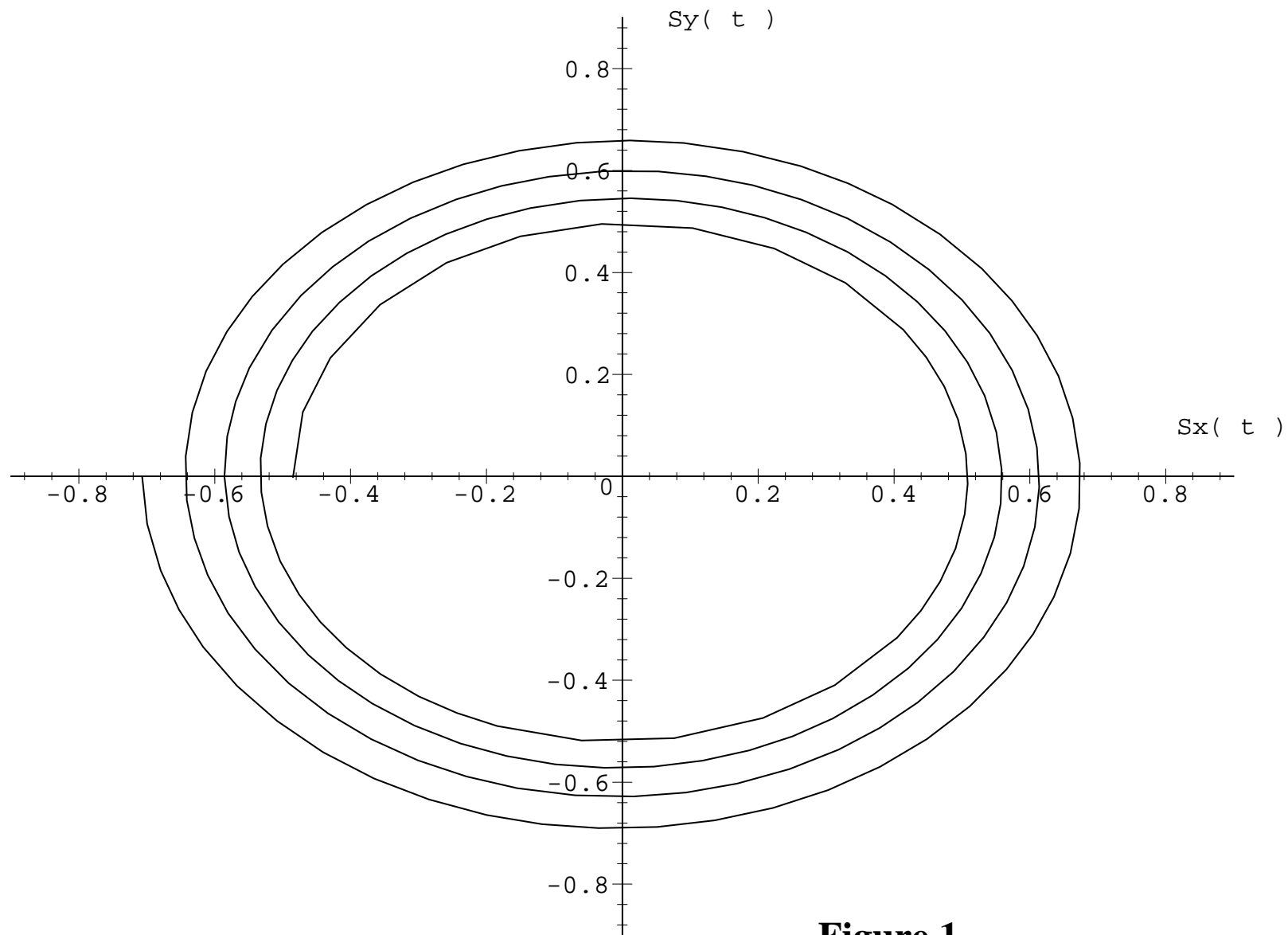
## References

- [1] M.V. Berry, *Proc. R. Soc.* **A392** (1984) 45.
- [2] J. Anandan, J. Christian and K. Wanelik, Resource Letter GPP-1: *Am. J. Phys.* **65** (1997) 180.
- [3] A.G. Wagh *et al.*, *Phys. Rev. Lett.* **78** (1997) 755.
- [4] C.L. Webb *et al.*, *Phys. Rev.* **A60** (1999) R1783.
- [5] J.C. Garrison and E.M. Wright, *Phys. Lett.* **A128** (1988) 177.
- [6] G. Dattoli, R. Mignani and A. Torre, *J. Phys.* **A23** (1990) 5795.
- [7] A. Kvitsinsky and S. Puttermann, *J. Math. Phys.* **32** (1991) 1403.
- [8] F. Gaitan, *Phys. Rev.* **A58** (1998) 1665.
- [9] D. Ellinas, S.M. Barnett and M.A. Dupertuis, *Phys. Rev.* **A39** (1989) 3228.
- [10] D. Gamliel and J.H. Freed, *Phys. Rev.* **A39** (1989) 3238.
- [11] W.H. Louisell, *Quantum Statical Properties of Radiation*, (Wiley, New York, 1973).
- [12] R. Kosloff, M.A. Ratner and W.B. Davies, *J. Chem. Phys.* **106** (1997) 7036.
- [13] G. Lindblad, *Comm. Math. Phys.* **39** (1974) 111.

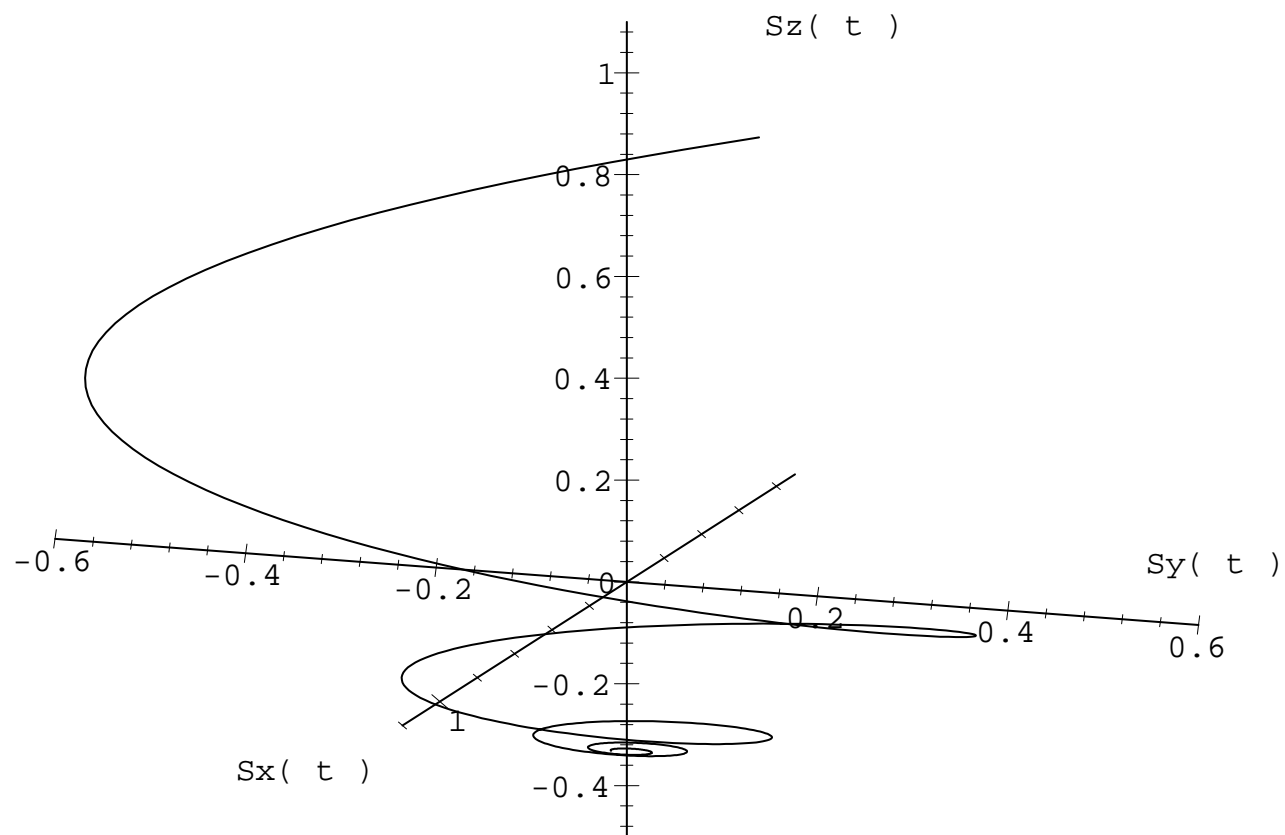
- [14] A. Müller, *Phys. Rev.* **A57** (1998) 731.
- [15] R.A. Harris and L. Stodolsky, *Phys. Lett.* **116B** (1982) 464.
- [16] L. Stodolsky, *Phys. Rev.* **D36** (1987) 2273.
- [17] G. Raffelt, G. Sigl and L. Stodolsky, *Phys. Rev. Lett.* **70** (1993) 2363.
- [18] J.I. Kim, M.C. Nemes, A.F.R. de Toledo Piza and H.E. Borges, *Phys. Rev. Lett.* **77** (1996) 207.
- [19] Like the case of no dissipation, the off-diagonal elements of the density matrix are not uniquely defined unless we impose a condition on  $\langle \phi_i; t | \left( \frac{d}{dt} | \phi_i; t \rangle \right)$ ,  $i = 1$  and  $2$ . In order to get eq.(12), we impose the conditions:  $\langle \phi_1; t | \left( \frac{d}{dt} | \phi_1; t \rangle \right) = i\omega \sin^2(\frac{\theta}{2})$  and  $\langle \phi_2; t | \left( \frac{d}{dt} | \phi_2; t \rangle \right) = -i\omega \sin^2(\frac{\theta}{2})$ .
- [20] M. Born und V. Fock, *Z. Phys.* **51** (1928) 165.
- [21] Chang-Pu Sun, *J. Phys.* **A21** (1988) 1595.
- [22] S. Hassani, *Mathematical Physics - A Modern Introduction to Its Foundations* (Springer, N.Y., 1999).
- [23] R.V. Churchill, *Fourier Series and Boundary Value Problems* (International Student Edition, Mc Graw-Hill, 1963).
- [24] If we make another choice for the instantaneous eigenstates of hamiltonian,  $|\Phi_i; t\rangle = e^{i\beta_i(t)}|\phi_i; t\rangle$ ,  $i = 1$  and  $2$ , the dynamics of the coefficient  $\alpha_{jl}^{(\beta)}(t)$  in the new basis is related to to the dynamics of  $\alpha_{jl}(t)$  (from eqs. (27)) by  $\frac{d\alpha_{jl}^{(\beta)}(t)}{dt} = e^{-i(\beta_j(0)-\beta_l(0))} \frac{d\alpha_{jl}(t)}{dt}$ .
- [25] H. Carmichael, *An Open Systems Approach to Quantum Optics, Lecture Notes in Physics* **m18** (Springer-Verlag, Berlin, 1993).
- [26] Private communication of M.C. Nemes and J.G. Peixoto de Faria.

Figure 1: Plot of the projection of Bloch's vector in the  $xy$  plane. We take  $\alpha = 0$ ,  $\theta = \frac{\pi}{4}$ ,  $\omega/\mu B = 10^{-3}$  and  $k/\mu B = 10^{-2}$ . Time evolution is plotted in the interval  $t = [0, \frac{4\pi}{\mu B}]$ .

Figure 2: Curve described by Bloch's vector in space. We take:  $\alpha = 0$ ,  $\theta = \frac{\pi}{4}$ ,  $\omega/\mu B = 10^{-3}$  and  $k/\mu B = 2.10^{-4}$ . The vector starts with positive  $z$ -component but ends up on the negative  $z$ -axis, after a time interval equal to the period of the external magnetic field.



**Figure 1**



**Figure 2**